Generalizing the Shift Method for Rectangular Shaped Vertices with Visibility Constraints*

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Abstract. In this paper we present a generalization of the *shift method* algorithm [4, 6] to obtain a straight-line grid drawing of a triconnected graph, where vertex representations have a certain specified size. We propose vertex representations having a rectangular shape. Additionally, one may demand maintainance of the criterion of strong visibility, that is, any possible line segment connecting two adjacent vertices cannot cross another vertex' representation. We prove that the proposed method produces a straight-line grid drawing of a graph in linear time with an area bound, that is only extended by the size of the rectangles, compared to the bound of the original algorithm.

1 Introduction

The shift method [4] is a well-known method among several approaches to obtain a standard straight-line representation of planar graphs in the graph drawing literature [2, 7, 9]. Given a triangulated graph, the original algorithm calculates coordinates for each vertex on an 2D integer grid such that the final drawing has a quadratic area bound. A linear time variant is presented in [3], [6] provides a version for triconnected graphs, [5] for biconnected graphs.

The approach presented in the following sections is related to a version of the shift method given in [1], which allows square vertex representations. In this paper, the shift method for triconnected graphs [6] is generalized to have rectangular shaped vertex representations. Furthermore, we demand that the criterion of strong visibility between adjacent vertices is satisfied, that is, any possible line segment connecting two adjacent vertices does not cross another vertex' representation. To maintain the strong visibility criterion in the shift method, additional shifts have to be introduced. The main contribution is to prove that the proposed method produces a grid drawing with an area quadratic in the sum of number of vertices and the sizes of the vertex representations.

The generalized shift method can be used to draw clustered graphs having planar quotient graphs [8]. Other possible applications include drawing graphs that have arbitrary vertex representations by using the minimal bounding box,

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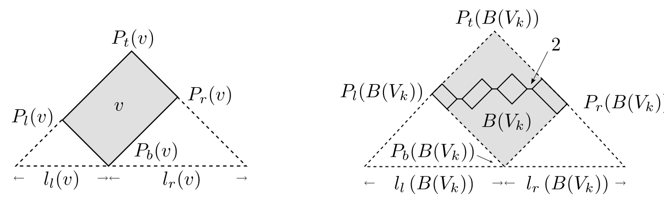

Fig. 1: Vertex representations. Left: singleton $V_k = \{v\}$. Right: $|V_k| > 1$.

or drawing graphs with labeled vertices, where the positions of a vertex and its label are not known, but only the size of the region into which they are allowed to be drawn.

2 Preliminaries

Let $G = (V, E)$ be a graph with $n = |V|$ and $m = |E|$. A graph is called *planar* if it has a crossing-free drawing in the plane. A *plane graph* is a planar graph with a fixed cyclic ordering of edges incident to each vertex and a fixed outer face. A plane graph divides the plane into which it is drawn into connected regions called *faces*. A *triconnected* graph is a graph where the removal of any pair of vertices does not disconnect the graph.

Let G be a triconnected plane graph. Let $\pi = (V_1, V_2, \ldots, V_K), K < n$, be a *lmc-ordering* of G as presented in [6]. It is shown that every triconnected plane graph has a *lmc*-ordering, and it can be computed in linear time. Let $G_k, k \leq K$, be the graph induced by $V_1 \cup \cdots \cup V_k$ according to π, particularly $G_K = G$. We denote by $C_0(G_k)$ the boundary of the outer face of G_k.

Vertices are represented as rectangles rotated by 45 degrees. For all $v \in V$, vertex lengths $l_l(v)$ and $l_r(v)$ are given according to the side lengths of a vertex representation, as illustrated in Fig.1. Let $l(v) = l_l(v) + l_r(v)$. Let $P_l(v)$, $P_r(v)$, $P_b(v)$ and $P_t(v)$ be the left, right, bottom and top corners of v's representation, with $P_l(v) = (x_l(v), y_l(v))$, etc. As illustrated in Fig.1, we represent a set $V_k = \{v_k^1, \ldots, v_k^j\}, j > 1$, as a chain of the single vertices, where $[P_r(v_k^i), P_l(v_k^{i+1})], 1 \leq i < j$, are horizontally aligned with distance two. Let $l(V_k) = \sum_{v \in V_k} l(v)$, $l_l(V_k) = \sum_{v \in V_k} l_l(v)$, and $l_r(V_k)$ accordingly. Let $B(V_k)$ be the minimal bounding box of the representation of V_k. For a singleton $V_k = \{v_k\}$, the corner points of $B(V_k)$ are exactly the corner points of v_k. To obtain a grid drawing, we assume without loss of generality that $l_l(v), l_r(v) \in \mathbb{N}_0$ for all $v \in V$ and both are even.

For vertex representations having an area, as the representation given above, we can define the criterion of *strong visibility* for graph drawing algorithms:

Definition 1 (Strong visibility). *Let $v, w \in V$. Then v is* strongly visible *to w, if any line segment connecting a point within the representation of v to a point within the representation of w does not cross the representation of any other vertex $u \in V$ with $u \neq v, w$.*

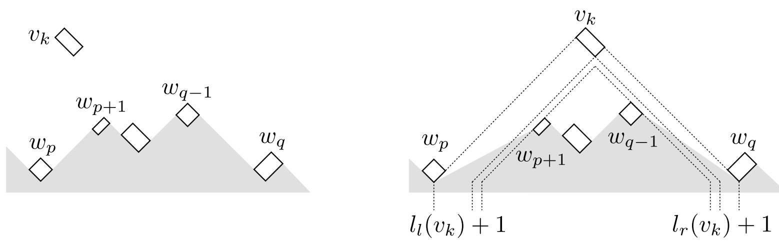

Fig. 2: Installing vertex v_k. Left: G_{k-1}. Right: G_k

Let P_1 and P_2 be two grid points on an integer grid and let $\mu(P_1, P_2)$ be the intersection point of the straight-line segment with slope $+1$ through P_1 and the straight-line segment with slope -1 through P_2. In the algorithm, vertices will be placed according to μ; hence the rotation of vertex representations by 45 degrees. Let $L(v)$ be a set of *dependent* vertices of v, that will later on contain the vertices which have to be rigidly moved with v when v itself is moved.

3 Algorithm

The algorithm starts by drawing G_2. We place $V_1 = \{v_1^1, v_1^2\}$ and V_2 with coordinates $P_r(v_1^1) \leftarrow (0,0)$, $P_l(v_1^2) \leftarrow (l(V_2) + \max\{l_l(v_1^1), l_r(v_1^2)\} + 2 \cdot |V_2|, 0)$ and $P_t(B(V_2)) \leftarrow \mu\left(P_r\left(v_1^1\right), P_l\left(v_1^2\right)\right)$. The sets of dependent vertices are initialized with $L(v) \leftarrow \{v\}$ for $v \in G_2$. We proceed by placing the next set V_k in the *lmc*-ordering into G_{k-1}, one by one, starting with V_3. Let $C_0\left(G_{k-1}\right) = w_1, \ldots, w_t$, $w_1 = v_1$ and $w_t = v_2$. Assume that following conditions hold for G_{k-1}, $k \geq 3$:

(C1) $x_r\left(w_i\right) < x_l\left(w_{i+1}\right)$, $1 \leq i \leq t-1$.
(C2) each straight-line segment $\left(P_r\left(w_i\right), P_l\left(w_{i+1}\right)\right)$, $1 \leq i \leq t-1$, has either slope $+1$, 0 or -1.
(C3) every vertex in G_{k-1} is strongly visible to its adjacent vertices in G_{k-1}.

Obviously, these conditions hold for the initial Graph G_2. When inserting V_k, let $w_1, \ldots, w_p, w_{p+1}, \ldots, w_q, \ldots, w_t$ be the vertices on $C_0(G_{k-1})$, where w_p is the leftmost and w_q the rightmost adjacent vertex of V_k in G_{k-1}. Similar to [3, 6], install $V_k = \{v_k^1, \ldots, v_k^j\}$ by applying the following steps, see Fig.2.

Step 1. for all $v \in \bigcup_{i=p+1}^{q-1} L(w_i)$ do $x(v) \leftarrow x(v) + l_l(V_k) + |V_k|$
Step 2. for all $v \in \bigcup_{i=q}^{t} L(w_i)$ do $x(v) \leftarrow x(v) + l_l(V_k) + l_r(V_k) + 2 \cdot |V_k| + \Delta$
Step 3. $P_t(B(V_k)) \leftarrow \mu\left(P_r\left(w_p\right), P_l\left(w_q\right)\right)$
Step 4. For one $j', 1 \leq j' \leq j$ set $L(v_k^{j'}) \leftarrow \left\{v_k^{j'} \cup \left(\bigcup_{i=p+1}^{q-1} L(w_i)\right)\right\}$;
for all other $j'' \neq j', 1 \leq j'' \leq j$ set $L(v_k^{j''}) \leftarrow \{v_k^{j''}\}$

Actually, if V_k is not a singleton, the bottom corner of $B(V_k)$ is placed too low by $|V_k| - 1$. Nevertheless, this is sufficient since every vertex in V_k is separated by distance two, and therefore the lowest possible bottom corner of any $v \in V_k$ is at least $|V_k| - 1$ higher than $P_b(B(V_k))$. Assume for the moment that $\Delta = 0$ in step 2. Then all conditions are satisfied for G_k if $\{w_{p+1}, \ldots, w_{q-1}\} \neq \emptyset$, see [8]. However, if there are *no inner vertices* between w_p and w_q on the outer face of G_{k-1}, and $l_l(w_p), l_r(w_q) \neq 0$, condition (C3) is violated in G_k by placing V_k in steps 1 to 4, as w_q is not strongly visible to w_p anymore after insertion. Since step 1 will be omitted in this case, the problem can only be addressed by introducing an extra shift Δ in step 2, thus placing V_k high enough in step 3 such that the strong visibility between w_p and w_q is not violated in G_k. The following Lemma shows how much extra shift is needed, when installing V_k.

Lemma 1. *Let $V_k = \{v_k\}$. Let $\{w_{p+1}, \ldots, w_{q-1}\} = \emptyset$ and $l_l(w_p), l_r(w_q) \neq 0$. Then w_p will be strongly visible to w_q in G_k, if an extra shift amount Δ is added in step 2 with*

$$\Delta = \begin{cases} \left\lceil 2 \cdot \frac{l_l(w_p) \cdot l_r(w_q)}{l_l(w_p) + l_l(w_q) + l_r(w_q)} \right\rceil & \textit{if } [P_r(w_p), P_l(w_q)] \textit{ has slope } +1 \textit{ in } G_{k-1} \\ \left\lceil 2 \cdot \frac{l_l(w_p) \cdot l_r(w_q) - 4}{l_l(w_p) + l_r(w_q) + 4} \right\rceil & \textit{if } [P_r(w_p), P_l(w_q)] \textit{ has slope } 0 \textit{ in } G_{k-1} \\ \left\lceil 2 \cdot \frac{l_l(w_p) \cdot l_r(w_q)}{l_l(w_p) + l_r(w_p) + l_r(w_q)} \right\rceil & \textit{if } [P_r(w_p), P_l(w_q)] \textit{ has slope } -1 \textit{ in } G_{k-1} \end{cases}$$

Proof. Let δ be the height, with which v_k must be lifted upwards to guarantee strong visibility. Assume $[P_r(w_p), P_l(w_q)]$ has slope $+1$ in G_{k-1}, as illustrated in Fig.3 (left). The gray rectangle indicates the position of v_k in G_k without introducing an extra shift. Let $\delta_{pq} = \sqrt{2} \cdot \overline{[P_r(w_p), P_l(w_q)]}$. Observe that δ is largest, if δ_{pq} has the smallest possible value, and that at the same time $\delta_{pq} \geq l_l(w_q)$. Thus, assume $\delta_{pq} = l_l(w_q)$. By the theorem on intersecting lines, we have

$$\frac{\delta}{l_l(w_p)} = \frac{l_r(w_q)}{l_l(w_p) + l_l(w_q) + l_r(w_q)} \quad \Leftrightarrow \quad \delta = \frac{l_l(w_p) \cdot |w_q|r}{l_l(w_p) + l_l(w_q) + l_r(w_q)}$$

It is easy to see that δ is analogous, if the line segment $[P_r(w_p), P_l(w_q)]$ has slope -1 in G_{k-1}. Assume $[P_r(w_p), P_l(w_q)]$ has slope 0 in G_{k-1}, as shown in Fig.3 (right). In this case, $P_r(w_p)$ and $P_l(w_q)$ are separated by a horizontal line segment with length two. Assume that $l_l(w_p) < l_r(w_q)$, then

$$\begin{aligned} \delta + 1 &= \tfrac{l_l(w_p)}{2} + \tfrac{l_l(w_p)/2 \,+\, 1}{l_l(w_p)/2 + 2 + l_r(w_q)/2} \cdot \tfrac{l_r(w_q) - l_l(w_p)}{2} \\ \Leftrightarrow \quad \delta &= \tfrac{l_l(w_p) \cdot l_r(w_q) - 4}{l_l(w_p) + l_r(w_q) + 4} \end{aligned}$$

The same value is obtained, if $l_l(w_p) \geq l_r(w_q)$. Overall, if an extra shift $\Delta = \lceil 2\delta \rceil$ is introduced, v_k is lifted by at least δ, and hence w_p and w_q will be strongly visible to each other in G_k. □

Observe that, if V_k is not a singleton, we have to add $2 \cdot (|V_k| - 1)$ to Δ, since $P_b(B(V_k))$ is $|V_k| - 1$ lower than the bottom corner of a singleton v_k, as indicated in Fig.3. Note also that, if Δ is an odd number, it has to be increased by one to maintain the grid drawing property.

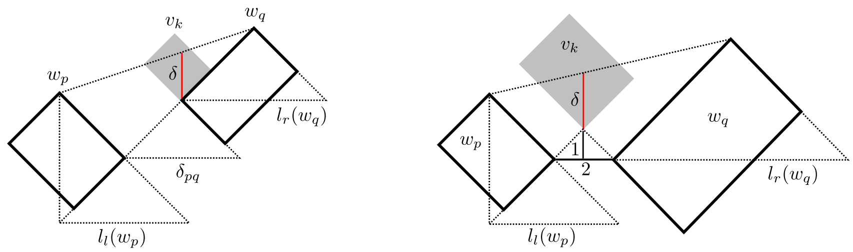

Fig. 3: Geometry for the case $\{w_{p+1}, \dots, w_{q-1}\} = \emptyset$. Left: slope +1. Right: slope 0.

4 Analysis

The following theorems state the bounds for the drawing area of the proposed method, and its time complexity.

Theorem 1. *The total grid area of a drawing of a triconnected plane graph $G = (V, E)$ with given vertex lengths $l_l(v), l_r(v), v \in V$ produced by the proposed method is in $O\left(|V| + \sum_{v \in V} l(v)\right)^2$.*

Proof. The width of the initial layout of G_2 is clearly bounded by $2 \cdot |V_2| + \Delta_2 + \sum_{i=1}^{2} l(V_i)$, with $\Delta_2 = \max(l_l(v_1^1), l_r(v_1^2))$. Whenever a set V_k is added, the width increases by $2 \cdot |V_k| + \Delta_k + l(V_k)$, where Δ_k denotes the extra shift in step k. Thus, the total width is bounded by $2 \cdot |V| + \sum_{v \in V} l(v) + \sum_{i=2}^{K} \Delta_i$.

Assume that all $V_k, 2 < k \leq K$, are singleton, and that, instead of shifting exactly with $\Delta = \lceil 2\delta \rceil$ when installing V_k, we shift with either $\max(l_l(w_p), l_r(w_q))$ or $\min(l_l(w_p), l_r(w_q))$. If $[P_r(w_p), P_l(w_q)]$ has slope +1 in G_{k-1} and

$$\begin{array}{llcllcl} 1. & l_l(w_p) \geq \delta_{pq} + l_r(w_q) & & \delta \leq l_l(w_p)/2, & & \Delta = l_l(w_p) \\ 2. & l_r(w_q) \leq l_l(w_p) < \delta_{pq} + l_r(w_q) & \Rightarrow & \delta \leq l_r(w_q)/2, & \text{then} & \Delta = l_r(w_q) \\ 3. & l_r(w_q) \geq l_l(w_p) + \delta_{pq} & & \delta \leq l_r(w_q)/2, & & \Delta = l_r(w_q) \\ 4. & l_l(w_p) < l_r(w_q) < l_l(w_p) + \delta_{pq} & & \delta \leq l_l(w_p)/2, & & \Delta = l_l(w_p) \end{array}$$

are sufficient to maintain strong visibility. If $[P_r(w_p), P_l(w_q)]$ has slope −1 in G_{k-1}, the bounds are analogous. If $[P_r(w_p), P_l(w_q)]$ has slope 0, δ is bounded by $\max(l_l(w_p), l_r(w_q))/2$, therefore we assume to shift with the maximum length in this case. To find an upper bound for $\sum \Delta$ we use amortized analysis.

Consider the part of $\sum \Delta$ which is contributed due to shifting with the maximum length of $l_l(w_p)$ and $l_r(w_q)$, i.e. cases 1 and 3, and the case where the slope of $[P_r(w_p), P_l(w_q)]$ is 0. It is easy to see that, after one of these cases occured on one side of a vertex v at step k, the length of v on the same side only contributes to another extra shift at step $k' > k$ as the minimum length of the two adjacent vertices of $V_{k'}$. Hence, this part of $\sum \Delta$ is bounded by $\sum_{v \in V} l(v)$.

For determining the part of $\sum \Delta$ which is contributed due to shifting with the minimum length, let each vertex v have two amounts *left*(v) and *right*(v), that it can spend to support one extra shift on its left side and one on its right side. Set *left*$(v) \leftarrow l_r(v)$ and *right*$(v) \leftarrow l_l(v)$. Let w_p and w_q be the neighbors of V_k on the outer face of G_{k-1} at step k with $\{w_{p+1}, \ldots, w_{q-1}\} = \emptyset$. Assume $[P_r(w_p), P_l(w_q)]$ has slope $+1$ in G_{k-1}. Since in this case w_q was inserted later than w_p, it cannot have spent *left*(w_q), because otherwise there would be an inner vertex between w_p and w_q on the outer face. If $\min\{l_l(w_p), l_r(w_q)\} = l_r(w_q)$, then w_q pays for the extra shift with *left*(w_q). Suppose now that $\min\{l_l(w_p), l_r(w_q)\} = l_l(w_p)$. If w_p has not used *right*(w_p) so far, then it just pays for the shift. If on the other hand *right*(w_p) has already been spent (e.g. to insert w_q), then w_q uses *left*$(w_q) = l_r(w_q) \geq l_l(w_p)$ to pay the extra shift. The payment is analogous if $[P_r(w_p), P_l(w_q)]$ has slope -1 in G_{k-1}. Thus, the total amount of extra shift is sufficiently paid, and this part of $\sum \Delta$ is therefore also bounded by $\sum_{v \in V} l(v)$. The additional amount of extra shift which is contributed, if V_k are not singleton, is clearly bounded by $2 \cdot \sum_{2 \leq i \leq K}(|V_k| - 1) < 2 \cdot |V|$.

Since $G = G_K$ satisfies condition (c2), the height of the drawing is bounded by half of its width plus the part of vertices v_1^1 and v_1^2 beneath the x-axis. □

If the strong visibility constraint has not to be maintained, the drawing area is exactly $\left(\frac{l(v_1^1)+l(v_1^2)}{2} + 2\omega\right) \times \left(\frac{\max(l_r(v_1^1), l_l(v_1^2))}{2} + \omega\right)$, $\omega = |V| - 2 + \sum_{i=2}^{K} \frac{l(V_i)}{2}$, since no extra shift is needed in this case. It remains an open problem to give a worst-case scenario and sharp area bound if strong visibility has to be guaranteed.

The linear time implementation of the original shift method [3] can easily be extended to our problem. Since the determination of the extra shift amount takes only constant time, the overall asymptotic complexity is not changed.

Theorem 2. *Given a triconnected plane graph $G = (V, E), n = |V|$, the proposed method can be implemented with running time $O(n)$.*

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