On the Recognition of Fan-Planar and Maximal Outer-Fan-Planar Graphs ⋆

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Abstract. *Fan-planar* graphs were recently introduced as a generalization of 1-*planar* graphs. A graph is *fan-planar* if it can be embedded in the plane, such that each edge that is crossed more than once, is crossed by a bundle of two or more edges incident to a common vertex. A graph is *outer-fan-planar* if it has a fan-planar embedding in which every vertex is on the outer face. If, in addition, the insertion of an edge destroys its outer-fan-planarity, then it is *maximal outer-fan-planar*.

In this paper, we present a polynomial-time algorithm to test whether a given graph is *maximal outer-fan-planar*. The algorithm can also be employed to produce an outer-fan-planar embedding, if one exists. On the negative side, we show that testing fan-planarity of a graph is NP-hard, for the case where the *rotation system* (i.e., the cyclic order of the edges around each vertex) is given.

1 Introduction

A *simple drawing* of a graph is a representation of a graph in the plane, where each vertex is represented by a point and each edge is a Jordan curve connecting its endpoints such that no edge contains a vertex in its interior, no two edges incident to a common end-vertex cross, no edge crosses itself, no two edges meet tangentially, and no two edges cross more than once.

An important subclass of drawn graphs is the class of planar graphs, in which there exist no crossings between edges. Although planarity is one of the most desirable properties when drawing a graph, many real-world graphs are in fact non-planar.

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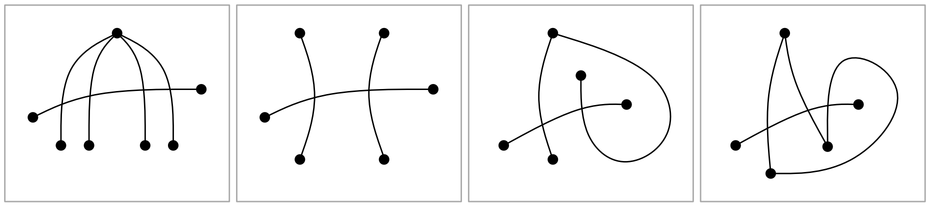


(a) Fan-crossing (b) Forbidden pattern I (c) Forbidden pattern II (d) Triangle crossing

Fig. 1. (taken from [18]) (a) A fan-crossing. (b) Forbidden crossing pattern I: An edge cannot be crossed by two independent edges. (c) Forbidden crossing pattern II: An edge cannot be crossed by two edges having their common end-point on different sides of it. (d) Forbidden crossing pattern II implies that an edge cannot be crossed by three edges forming a triangle.

On the other hand, it is accepted that edge crossings have negative impact on the human understanding of a graph drawing [22] and simultaneously it is NP-complete to find drawings with minimum number of edge crossings [14]. This motivated the study of "almost planar" graphs which may contain crossings as long as they do not violate some prescribed forbidden crossing patterns. Typical examples include k-planar graphs [23], k-quasi planar graphs [2], RAC graphs [9] and fan-crossing free graphs [7].

Fan-planar graphs were recently introduced in the same context [18]. A *fan-planar drawing* of graph $G = (V, E)$ is a simple drawing which allows for more than one crossing on an edge $e \in E$ iff the edges that cross e are incident to a common vertex on the same side of e. Such a crossing is called *fan-crossing*. An equivalent definition can be stated by means of forbidden crossing patterns; see Fig. 1. A graph is *fan-planar* if it admits a fan-planar drawing. So, the class of fan-planar graphs is in a sense the complement of the class of fan-crossing free graphs [7], which simply forbid fan-crossings.

Kaufmann and Ueckerdt [18] showed that a fan-planar graph on n vertices cannot have more than $5n - 10$ edges; a tight bound. An *outer-fan-planar drawing* is a fan-planar drawing in which all vertices are on the outer-face. A graph is *outer-fan-planar* if it admits an outer-fan-planar drawing. An outer-fan-planar graph is *maximal outer-fan-planar* if adding any edge to it yields a graph that is not outer-fan-planar. Note that the forbidden pattern II is irrelevant for outer-fan-planarity.

Our main contribution is a polynomial time algorithm for the recognition of maximal outer-fan-planar graphs (Section 2). We also prove that the general fan-planar problem is NP-hard, for the case where the *rotation system* (i.e., the circular order of the edges around each vertex) is given (Section 3). Due to space restrictions, some proofs are omitted or only sketched in the text; full proofs for all results can be found in [5].

Related Work: As already stated, k-planar graphs [23], k-quasi planar graphs [2], RAC graphs [9] and fan-crossing free graphs [7] are closely related to the class of graphs that we study. A graph is *k-planar*, if it can be embedded in the plane with at most k crossings per edge. Obviously, 1-planar graphs are also fan-planar. A 1-planar graph with n vertices has at most $4n-8$ edges and this bound is tight [21]. Grigoriev and Bodlaender [15], and, independently Kohrzik and Mohar [19]) proved that the problem of determining whether a graph is 1-planar is NP-hard and remains NP-hard, even if the deletion of an edge makes the input graph planar [6]. On the positive side, Eades et al. [10] presented a linear time algorithm for testing *maximal* 1-planarity of graphs with a given rotation system. Testing outer-1-planarity of a graph can be solved in linear time, as shown independently by Auer et al. [4] and Hong et al. [17]. In addition, every outer-1-planar graph admits an outer-1-planar straight-line drawing [12]. Note that an outer-1-planar graph is always planar [4], while this is not true in general for outer-fan-planar graphs. Indeed, the complete graph K_5 is outer-fan-planar, but not planar.

A drawn graph is k*-quasi planar* if it has no k mutually crossing edges. It is conjectured that the number of edges of a k-quasi planar graph is linear in the number of its vertices. Pach et al. [20] and Ackerman [1] affirmatively answered this conjecture for 3- and 4-quasi planar graphs. Fox and Pach [13] showed that a k-quasi-planar n-vertex graph has at most $O(n \log^{1+o(1)} n)$ edges. Fan planar graphs are 3-quasi planar [18].

A different forbidden crossing pattern arises in RAC drawings where two edges are allowed to cross, if the crossings edges form right angles. Graphs that admit such drawings (with straight-line edges) are called *RAC graphs*. Didimo et al. [9] showed that a RAC graph with n vertices has no more than $4n - 10$ edges; a tight bound. RAC graphs are quasi planar [9], while maximally dense RAC graphs (i.e., RAC graphs with n vertices and exactly $4n - 10$ edges) are 1-planar [11]. Testing whether a graph is RAC is NP-hard [3]. Dekhordi and Eades [8] proved that every outer-1-plane graph has a straight-line RAC drawing, at the cost of exponential area.

Preliminaries: Unless otherwise specified, we consider finite, undirected, simple graphs. We also assume basic familiarity with SPQR-trees [16] (a short introduction is given in [5]). The *rotation system* of a drawing is the counterclockwise order of the incident edges around each vertex. The *embedding* of a drawn graph consists of its rotation system and for each edge the sequence of edges crossing it. For a graph G and a vertex $v \in V[G]$, we denote by $G - \{v\}$ the graph that results from G by removing v.

Lemma 1. *A biconnected graph G is outer-fan-planar if and only if it admits a straight-line outer-fan-planar drawing in which the vertices of G are restricted on a circle $\mathcal{C}$.*

Sketch of Proof. Let G be an outer-fan-planar graph and let Γ be an outer-fan-planar drawing of G. We will only show that G has a straight-line outer-fan-planar drawing whose vertices lie on a circle $\mathcal{C}$ (the other direction is trivial). The order of the vertices along the outer face of Γ completely determines whether two edges cross, as in a simple drawing no two incident edges can cross and any two edges can cross at most once. Now, assume that two edges cross another edge in Γ. Then, both edges have to be incident to the same vertex; hence, cannot cross each other. So, the order of the crossings on an edge is also determined by the order of the vertices on the outer face. Therefore, we can construct a drawing $\Gamma_{\mathcal{C}}$ by placing the vertices of G on a circle $\mathcal{C}$ preserving their order in the outer face of Γ and draw the edges as straight-line segments. □

2 Recognizing and Drawing Maximal Outer-Fan-Planar Graphs

In this section, we prove that given a graph $G = (V, E)$ on n vertices, there is a polynomial time algorithm to decide whether G is maximal outer-fan-planar and if so a corresponding straight-line drawing can be computed in linear time. By Lemma 1, we only have to check, whether G has a straight-line drawing on a circle $\mathcal{C}$ that is fan-planar. Note that such a drawing is determined by the cyclic order of the vertices on $\mathcal{C}$. Since fan-planar graphs with n vertices have at most $5n - 10$ edges [18], we may assume that the number of edges is linear in the number of vertices. We first consider the case that G is 3-connected and then using SPQR-trees we show how the problem can be solved for biconnected graphs. Observe that biconnectivity is a necessary condition for maximal outer-fan-planarity. Indeed, if an outer-fan-planar drawing has a cut-vertex c, it is easy to see that it is always possible to draw an edge connecting two neighbors of c while preserving the outer-fan-planarity.

The 3-Connected Case: Assume that a straight-line drawing of a 3-connected graph G with n vertices on a circle $\mathcal{C}$ is given. Let $v_1, \ldots, v_n$ be the order of the vertices around $\mathcal{C}$. An edge $\{v_i, v_j\}$ is an *outer edge*, if $i - j \equiv \pm 1 \pmod{n}$, a *2-hop*, if $i - j \equiv \pm 2 \pmod{n}$, and a *long edge* otherwise. G is a *complete 2-hop graph*, if there are all outer edges and all 2-hops, but no long edges. Two crossing long edges are a *scissor* if their end-points form two consecutive pairs of vertices on $\mathcal{C}$. We say that a triangle is an *outer triangle* if two of its three edges are outer edges. We call an outer-fan-planar drawing *maximal*, if adding any edge to it yields a drawing that is not outer-fan-planar.

Our algorithm is based on the observation that if a graph is 3-connected maximal outer-fan-planar, then it is a complete 2-hop graph, or we can repeatedly remove any degree-3 vertex from any 4-clique until only a triangle is left. In a second step, we reinsert the vertices maintaining outer-fan-planarity (if possible). It turns out that we have to check a constant number of possible embeddings. In the following, we prove some necessary properties. The first three lemmas are used in the proof of Lemma 5. Their proofs are based on the 3-connectivity of the input graph; see Fig. 2a, 2b and 2c.

Lemma 2. *Let G be a 3-connected outer-fan-planar graph embedded on a circle $\mathcal{C}$. If two long edges cross, then two of its end-points are consecutive on $\mathcal{C}$.*

Lemma 3. *Let G be a 3-connected outer-fan-planar graph embedded on a circle $\mathcal{C}$. If there are two long crossing edges, then there is a scissor, as well.*

Lemma 4. *Let G be a 3-connected graph embedded on a circle $\mathcal{C}$ with a maximal outer-fan-planar drawing. If G contains a scissor, then its end vertices induce a K_4.*

Lemma 5. *Let G be a 3-connected graph with a maximal outer-fan-planar drawing and assume that the drawing contains at least one long edge. Then, G contains a K_4 with all four vertices drawn consecutively on the circle.*

Proof. First consider the case where the graph contains at least two crossing long edges and, thus, by Lemma 3 a scissor. Removing the vertices of a scissor, splits G into two connected components. Assume that we have chosen the scissor such that the smaller

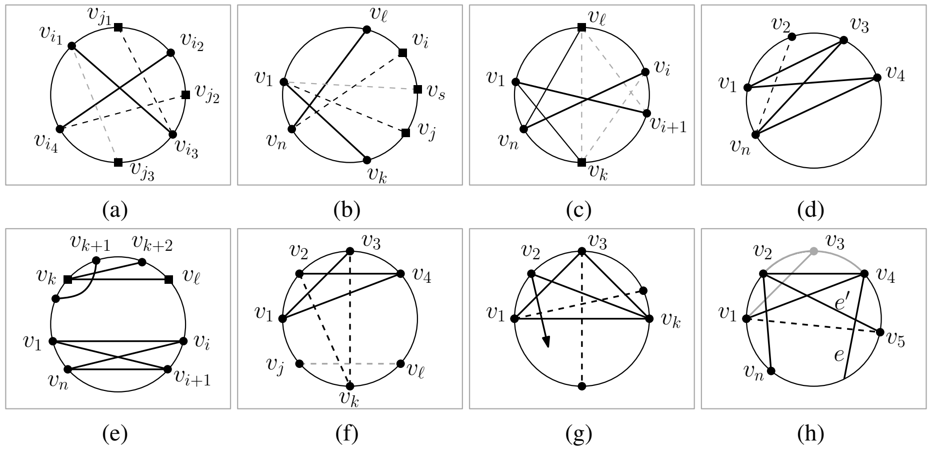


Fig. 2. Different configurations used in: (a) Lemma 2, (b) Lemma 3, (c, d) Lemma 4, (e) Lemma 5, (f) Lemma 6, (g) Lemma 7, (h) Lemma 9.

of the two components is as small as possible (thus, scissor-free) and that the vertices around $\mathcal{C}$ are labeled such that this scissor is $\{v_1, v_{i+1}\}$, $\{v_i, v_n\}$ with $i \leq n - i$, i.e., the component induced by $v_2, \ldots, v_{i-1}$ is the smaller one. Recall that by Lemma 4 a scissor induces a K_4.

If $i = 3$, i.e., if $\{v_1, v_3\}$ is a 2-hop, then G should contain either $\{v_2, v_n\}$ or $\{v_2, v_4\}$, as otherwise v_1 and v_3 is a separation pair; see Fig. 2d. Say w.l.o.g. $\{v_2, v_n\}$. Then, v_1, v_2, v_3 together with v_n induce a K_4 with all vertices consecutive on circle $\mathcal{C}$.

If $i > 3$, let $\{v_k, v_\ell\}$, $1 \leq k < \ell \leq i$ be a long edge such that there is no long edge $\{v_{k'}, v_{\ell'}\} \neq \{v_k, v_\ell\}$ with $k \leq k' < \ell' \leq \ell$; see Fig. 2e. Then, no long edge is crossing the edge $\{v_k, v_\ell\}$, as otherwise by Lemma 3 such a crossing would yield a new scissor, contradicting the choice of $\{v_1, v_{i+1}\}$ and $\{v_i, v_n\}$. Since $\{v_k, v_\ell\}$ is not crossed by a long edge, it must be crossed by exactly one 2-hop, say $\{v_{k-1}, v_{k+1}\}$. Now, $\ell - k > 3$ is not possible, since we could add the edge $\{v_{k+1}, v_\ell\}$, which is long. Hence, $\ell - k = 3$ and by maximality of the outer-fan-planar drawing, $v_k, v_{k+1}, v_{k+2}, v_\ell$ induces a K_4 with all vertices consecutive on $\mathcal{C}$. Finally, if G contains no two crossing long edges, let $\{v_k, v_\ell\}$, $1 \leq k < \ell \leq n$ be a long edge such that there is no long edge $\{v_{k'}, v_{\ell'}\} \neq \{v_k, v_\ell\}$ with $k \leq k' < \ell' \leq \ell$. By the same argumentation as above, we obtain that $v_k, v_{k+1}, v_{k+2}, v_\ell$ induces a K_4 with all vertices consecutive on $\mathcal{C}$. □

Lemma 6. *Let G be a 3-connected outer-fan-planar graph with at least six vertices. If G contains a K_4 with all vertices drawn consecutively on circle $\mathcal{C}$, then this K_4 contains exactly one vertex of degree three and this vertex is neither the first nor the last of the four vertices.*

Proof. Let the vertices around circle $\mathcal{C}$ be labeled so that v_1, v_2, v_3, v_4 induce a K_4. Since v_1 and v_4 is not a separation pair, there is an edge between v_2 or v_3 and a vertex, say v_k, among $v_5, \ldots, v_n$. Hence, three out of the four vertices v_1, v_2, v_3 and v_4 have degree at least four; see Fig. 2f. If v_3 had a neighbor in $v_5, \dots, v_n$, then this could only be v_k, as otherwise $\{v_1, v_4\}$ would be crossed by two independent edges. Since G has at least 6 vertices, we assume w.l.o.g. that $k > 5$. Since v_4 and v_k is not a separation pair, there has to be an edge $\{v_\ell, v_m\}$ for some $4 < \ell < k$ and a $j \notin \{4, \dots, k\}$. But such an edge would not be possible in an outer-fan-planar drawing. □

Lemma 7. *Let G be a 3-connected outer-fan-planar graph with at least six vertices. If G contains a K_4 with a vertex of degree 3, then this K_4 has to be drawn consecutively on circle $\mathcal{C}$ in any outer-fan-planar drawing of G.*

Proof. Observe that any outer-fan-planar drawing of a K_4 contains exactly one pair of crossing edges. If two 2-hops cross, then all vertices of the K_4 are consecutive. If the K_4 contains two crossing long edges, then each of the vertices of the K_4 is incident to an outer edge not contained in the K_4; thus, has degree at least four. If a long edge and a 2-hop cross, assume that the vertices around $\mathcal{C}$ are labeled such that v_1, v_2, v_3, v_k induce a K_4 for some $5 \leq k < n$; see Fig. 2g. Since v_1, v_3 and v_k are incident to an outer edge not contained in the K_4, they have degree at least four. We claim that v_2 has degree at least four. Since v_3 and v_k is not a separation pair, there is an edge between a vertex among $v_4, \dots, v_{k-1}$ and v_2 or v_1 and an edge between a vertex among $v_{k+1}, \dots, v_n$ and v_2 or v_3. Choosing v_1 and v_3 in the first and second case respectively, yields two independent edges crossing $\{v_2, v_k\}$. So, v_2 is connected to a vertex outside K_4. □

Lemma 8. *Let G be a 3-connected graph with $n \geq 5$ vertices and let $v \in V[G]$ be a vertex of degree three that is contained in a K_4. Then, $G - \{v\}$ is 3-connected.*

Proof. Let a, b, c and d be four arbitrary vertices of $G - \{v\}$. Since G was 3-connected, there was a path P from a to b in $G - \{c, d\}$. Assume that P contains v. Since v is only connected to vertices that are connected to each other, there is also another path from a to b in $G - \{c, d\}$ not containing v. Hence, a and b cannot be a separation pair in $G - \{v\}$. Since a and b were arbitrarily selected, $G - \{v\}$ is 3-connected. □

Lemma 9. *Let G be a 3-connected graph with $n > 6$ vertices, let v_1, v_2, v_3 and v_4 be four vertices that induce a K_4, such that the degree of v_3 is three. Then, $G - \{v_3\}$ has a maximal outer-fan-planar drawing if G has a maximal outer-fan-planar drawing.*

Proof. Consider a maximal outer-fan-planar drawing of G on a circle $\mathcal{C}$ and let v_1, v_2, v_3, $v_4, \dots, v_n$ be the order of the vertices on $\mathcal{C}$ (recall Lemma 7). Assume to the contrary that after removing v_3, we could add an edge e to the drawing; see Fig. 2h. By Lemma 6, $\{v_3, v_1\}$ is the only edge incident to v_3 that crosses some edges of $G - \{v_3\}$. Hence, there must be an edge e' that is crossed by e and $\{v_3, v_1\}$. Since $\{v_3, v_1\}$ crosses only edges incident to v_2 that also cross $\{v_1, v_4\}$, it follows that e' has to be incident to v_2. Further, since $G - \{v_3\}$ plus e is outer-fan-planar it follows that e is incident to v_1 or v_4. Moreover, since G plus e is not outer-fan-planar it follows that e is incident to v_4.

Let i be maximal so that there is an edge $\{v_2, v_i\}$. If $i \neq n$, then v_1 and v_i is a separation pair: Any edge connecting $\{v_{i+1}, \dots, v_{n-1}\}$ to $\{v_2, v_3, \dots, v_{i-1}\}$ and not being incident to v_2 crosses $\{v_2, v_i\}$. But edges crossing $\{v_2, v_i\}$ can only be incident to v_1, a contradiction. Now, let $j > 4$ be minimum such that there is an edge $\{v_2, v_j\}$.

We claim that $j = 5$. If this is not the case, then similarly to the previous case v_4 and v_j would be a separation pair in $G - \{v_3\}$ plus e, which is not possible due to Lemma 8.

It follows that G has to contain edge $\{v_1, v_5\}$: Since G is outer-fan-planar, in G there cannot be an edge $\{v_4, v_k\}$ for some $k = 6, \ldots, n$, since it would cross $\{v_2, v_5\}$ which is crossed by $\{v_3, v_1\}$. So, $\{v_1, v_5\}$ crosses only edges incident to v_2 that are already crossed by $\{v_3, v_1\}$ and $\{v_4, v_1\}$. Hence, $\{v_1, v_5\}$ could be added to G without violating outer-fan-planarity; a clear contradiction. Since e and $\{v_2, v_n\}$ both cross $\{v_1, v_5\}$ it follows that $e = \{v_4, v_n\}$. But now, v_5 and v_n has to be a separation pair. □

Remark 1. Let G be a graph with 6 vertices containing a vertex v of degree three. Then G is maximal outer-fan-planar if and only if $G - \{v\}$ is a K_5 missing one of the edges that connects a neighbor of v to one of the other two vertices.

Lemma 10. *It can be tested in linear time whether a graph is a complete 2-hop graph. Moreover, if a graph is a complete 2-hop graph, then it has a constant number of outer-fan-planar embeddings and these can be constructed in linear time.*

Proof. Let G be an n-vertex graph. We test whether G is a complete 2-hop as follows. If $n \in \{4, 5\}$, then G is either K_4 or K_5. Otherwise, check first whether all vertices have degree four. If so, pick one vertex as v_1, choose a neighbor as v_2 and a common neighbor of v_1 and v_2 as v_3 (if no such common neighbor exists then G is not a complete 2-hop). Assume now that we have already fixed $v_1, \ldots, v_i$, $3 \leq i < n$. Test whether there is a unique vertex $v \in V \setminus \{v_1, \ldots, v_i\}$ that is adjacent to v_i and v_{i-1}. If so, set $v_{i+1} = v$. Otherwise reject. If we have fixed the order of all vertices check whether there are only outer edges and 2-hops. Do this for any possible choices of v_2 and v_3, i.e., for totally at most 6 choices. □

Remark 2. No degree 3 vertex can be added to an n-vertex complete 2-hop with $n \geq 5$.

We are now ready to describe our algorithm. If the graph is not a complete 2-hop graph, recursively try to remove a vertex of degree 3 which is contained in a K_4. If G is maximal outer-fan-planar, Lemmas 5 and 6 guarantee that such a vertex always exists in the beginning. Remark 2 guarantees that also in subsequent steps there is a long edge and, thus, Lemmas 8 and 9 guarantee that also in subsequent steps, we can apply Lemma 5 as long as we have at least six vertices. Remark 1 guarantees that we can also remove two more vertices of degree 3 ending with a triangle.

At this stage, we already know that if the graph is outer-fan-planar, it is indeed maximal outer-fan-planar. Either, we started with a complete 2-hop graph or we iteratively removed vertices of degree three yielding a triangle. Note that in the latter case we must have started with $3n - 6$ edges. On the other hand, if we apply the above procedure to an n-vertex 3-connected maximal outer-fan-planar graph, we get that the number of its edges is exactly $2n$ or $3n - 6$.

Finally, we try to reinsert the vertices in the reversed order in which we have deleted them. By Lemma 7, we can insert the vertex of degree three only between its neighbor, that is, there are at most two possibilities where we could insert the vertex. Lemma 11 guarantees that in total, we have to check at most four possible drawings for G.

Lemma 11. *When reinserting a sequence of degree 3 vertices starting from a triangle, at most the first two vertices have two choices where they could be inserted.*

Proof. Let H be a outer-fan-planar graph and let three consecutive vertices v_1, v_2, v_3 induce a triangle. Assume, we want to insert a vertex v adjacent to v_1, v_2, v_3. By Lemma 6, we have to insert v between v_1 and v_2 or between v_2 and v_3. Note that the edges that are incident to v_2 and cross $\{v_1, v_3\}$ are also crossed by an edge e incident to v. So, if there is an edge incident to v_2 that was already crossed twice before inserting v, this would uniquely determine whether e is incident to v_1 or v_3 and, thus, where to insert v.

We will now show that after the first insertion each relevant vertex is incident to an edge that is crossed at least twice. When we insert the first vertex we create a K_4. From the second vertex on, whenever we insert a new vertex, it is incident to an edge that is crossed at least twice. Also, after inserting the second degree 3 vertex, three among the four vertices of the initial K_4 are also incident to an edge that is crossed at least twice. The forth vertex of the initial K_4 is not the middle vertex of a triangle consisting of three consecutive vertices. It can only become such a vertex if its incident inner edges are crossed by a 2-hop. But then these inner edges are all crossed at least twice. □

Summarizing, we obtain the following theorem; in order to exploit this result in the biconnected case, it is also tested whether a prescribed subset (possibly empty) of edges can be drawn as outer edges.

Theorem 1. *Given a 3-connected graph G with a subset E' of its edge set, it can be tested in linear time whether G is maximal outer-fan-planar and has an outer-fan-planar drawing such that the edges in E' are outer edges. Moreover if such a drawing exists, it can be constructed in linear time.*

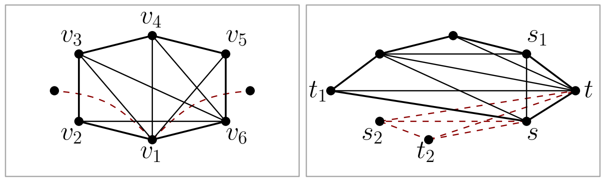
Sketch of Proof. Let n be the number of vertices. By Lemma 10, a complete 2-hop graph has only a constant number of outer-fan-planar embeddings which can be computed in linear time. Whenever we remove a vertex from the graph, we append it to a queue. Any vertex that was removed from the queue will never be appended again. Hence, there are at most n iterations.

To check whether the degree three vertices can be reinserted back in the graph, we only have to consider in total four different embeddings. Assume that we want to insert a vertex v into an outer triangle v_1, v_2, v_3. Then we just have to check whether v_1 or v_3 are incident to edges other than the edge $\{v_1, v_3\}$ that cross an edge incident to v_2. This can be done in constant time by checking only two pairs of edges. □

The Biconnected Case: We now sketch how to test outer-fan-planar maximality on a biconnected graph.

Lemma 12. *Let $v_1, \ldots, v_n$ be the order of the vertices around the circle in an outer-fan-planar drawing of a 3-connected graph G. If we can add a vertex v between v_1 and v_n with an edge $\{v, v_i\}$ for some $i = 2, \ldots, n-1$, then $i = 2$ or $i = n-1$.*

Proof. Otherwise, since v_1, v_i cannot be a separation pair of G, there has to be an edge from a v_k for some $k = 2, \ldots, i-1$ that crosses $\{v, v_i\}$ and hence an edge $\{v_k, v_n\}$. Since v_n, v_i cannot be a separation pair of G, there has to be an edge $\{v_1, v_\ell\}$ for some $\ell = i+1, \ldots, n-1$. But now there are three independent edges crossing. □



(a) (b)

Fig. 3. (a) In the solid graph, edge $\{v_2, v_3\}$ ($\{v_5, v_6\}$) is porous around v_2 (v_6, resp.). (b) Illustration of Case 5b of Theorem 2.

We say that an outer edge $\{v_1, v_n\}$ is *porous* around v_1 if we could add a vertex v between v_1 and v_n and an edge $\{v, v_2\}$ maintaining outer-fan-planarity. Note that any edge of a simple cycle, i.e., of the skeleton of an S-node is porous around any of its end vertices. Any outer edge of a K_4 is porous around any of its end vertices; see Fig. 3.

We use the SPQR-tree of a biconnected graph to characterize whether it is maximal outer-fan-planar; see [5] for a proof of this theorem.

Theorem 2. *A biconnected graph is maximal outer-fan-planar iff the following hold:*

1) The skeleton of any R-node is maximal outer-fan-planar and has an outer-fan-planar drawing in which all virtual edges are outer edges,
2) No R-node is adjacent to an R-node or an S-node,
3) All S-nodes have degree three,
4) All P-nodes have degree three and are adjacent to a Q-node, and
5) Let G_1 and G_2 be the skeleton of the two neighbors of a P-node other than the Q-node and let $\{s, t\}$ be the common virtual edge of G_1 and G_2. Then $G_i, i = 1, 2$ must not admit an outer-fan-planar drawing with t_i, s, t, s_i being consecutive around the circle and
   (a) edge $\{s, t\}$ is porous in both G_1 and G_2 around the same vertex, or
   (b) edge $\{t_1, s\}$ ($\{s_2, t\}$) is real and porous around s (t, resp.), or
   (c) edge $\{s_1, t\}$ ($\{t_2, s\}$) is real and porous around t (s, resp.).

As the number of outer-fan-planar embeddings of a 3-connected graph is bounded by a constant, the conditions of Thm. 2 can be tested in polynomial time. If the conditions are fulfilled, then an outer-fan-planar drawing can be constructed in linear time.

3 The NP-hardness of the FAN-PLANARITY WITH FIXED ROTATION SYSTEM Problem

In this section, we study the FAN-PLANARITY WITH FIXED ROTATION SYSTEM problem (FP-FRS), that is, the problem of deciding whether a graph $G = (V, E)$ with a fixed rotation system $\mathcal{R}$ admits a fan-planar drawing preserving $\mathcal{R}$.

Theorem 3. FAN-PLANARITY WITH FIXED ROTATION SYSTEM *is NP-hard.*

Proof. We prove the statement by using a reduction from 3-PARTITION (3P). An instance of 3P is a multi-set $A = \{a_1, a_2, \ldots, a_{3m}\}$ of $3m$ positive integers in the range $(B/4, B/2)$, where B is an integer such that $\sum_{i=1}^{3m} a_i = mB$. 3P asks whether A can be partitioned into m subsets $A_1, A_2, \ldots, A_m$, each of cardinality 3, such that the sum of the numbers in each subset is B. As 3P is *strongly* NP-hard [14], it is not restrictive to assume that B is bounded by a polynomial in m.

Before describing our transformation, we need to introduce the concept of *barrier gadget*. An n-vertex *barrier gadget* is a graph consisting of a cycle of $n \geq 5$ vertices plus all its 2-hop edges; a barrier gadget is therefore a maximal outer-2-planar graph. We make use of barrier gadgets in order to constraint the routes of some specific paths of G_A. Indeed, in a fan-planar drawing of a biconnected graph containing an outer-2-planar drawing Γ_b of a barrier gadget, no path can enter inside the boundary cycle of Γ_b and cross a 2-hop edge. Also, if a path enters in Γ_b without crossing any 2-hop edge, then it must immediately exit from Γ_b forming a fan-crossing with an outer edge of Γ_b.

Now, we are ready to describe how to transform an instance A of 3P into an instance $\langle G_A, \mathcal{R}_A \rangle$ of FP-FRS. We start from the construction of graph G_A which will be always biconnected. First of all, we create a *global ring barrier* by attaching four barrier gadgets G_t, G_r, G_b and G_l as depicted in Fig. 4. G_t is called the *top beam* and contains exactly $3mK$ vertices, where $K = \lceil B/2 \rceil + 1$. G_r is the *right wall* and has only five vertices. G_b and G_r are called the *bottom beam* and the *left wall*, respectively, and they are defined in a specular way. Observe that G_t, G_r, G_b and G_l can be embedded so that all their vertices are linkable to points within the closed region delimited by the global ring barrier. Then, we connect the top and bottom beams by a set of $3m$ *columns*, see Fig. 4 for an illustration of the case $m = 3$. Each *column* consists of a stack of $2m - 1$ *cells*; a *cell* consists of a set of pairwise disjoint edges, called the *vertical edges* of that cell. In particular, there are $m - 1$ *bottommost cells*, one *central cell* and $m - 1$ *topmost cells*. Cells of a same column are separated by $2m - 2$ barrier gadgets, called *floors*. Central cells (that are $3m$ in total) have a number of vertical edges depending on the elements of A. Precisely, the central cell C_i of the i-th column contains a_i vertical edges connecting its delimiting floors ($i \in \{1, 2, ..., 3m\}$). Instead, all the remaining cells have, each one, K vertical edges. Hence, a non-central cell contains more edges than any central cell. Further, the number of vertices of a floor is given by the number of its incident vertical edges minus two. Let u and v be the "central" vertices of the left and right walls, respectively (see also Fig. 4). We conclude the construction of graph G_A by connecting vertices u and v with m pairwise internally disjoint paths, called the *transversal paths* of G_A; each transversal path has exactly $(3m - 3)K + B$ edges.

Concerning the choice of a rotation system $\mathcal{R}_A$, we define a cyclic order of edges around each vertex that is compatible with the one depicted in Fig. 4. From what said, it is straightforward to see that an instance of 3P can be transformed into an instance of FP-FRS in polynomial time in m.

Let A be a *Yes*-instance of 3P, we show that $\langle G_A, \mathcal{R}_A \rangle$ admits a fan-planar drawing Γ_A preserving $\mathcal{R}_A$. We observe that such a drawing is easy to compute if one omits all the transversal paths. It is essentially a drawing like that one depicted in Fig. 4, where columns are one next to the other within the closed region delimited by the global ring barrier. However, by exploiting a solution $\{A_1, A_2, \ldots, A_m\}$ of 3P for the

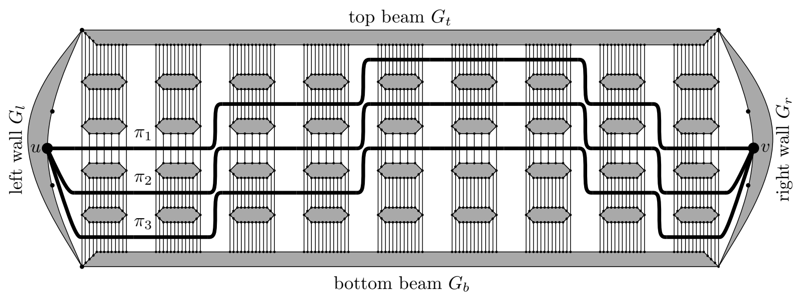


Fig. 4. Illustration of the reduction of FP-FRS from 3P, where $m = 3$, $B = 24$ and $A = \{7, 7, 7, 8, 8, 8, 8, 9, 10\}$. Transversal paths are routed according to the following solution of 3P: $A_1 = \{7, 7, 10\}$, $A_2 = \{7, 8, 9\}$ and $A_3 = \{8, 8, 8\}$.

instance A, also the transversal paths can be easily embedded without violating the fan-planarity. The idea is to route these paths in such a way that: (R.1) they do not cross each other; (R.2) they do not cross any barrier; (R.3) each path passes through exactly 3 central cells and $3m - 3$ non-central cells; (R.4) each cell is traversed by at most one path. Eventually, each transversal path crosses exactly $(3m - 3)K + B$ vertical edges, which is the same number of its edges. Therefore, it is possible to draw these paths by ensuring that each of their edges crosses exactly one vertical edge, which preserves the fan-planarity. Hence, eventually we get a fan-planar drawing Γ_A preserving the rotation system $\mathcal{R}_A$.

We conclude the proof by showing that if $\langle G_A, \mathcal{R}_A \rangle$ is a *Yes*-instance of FP-FRS, then A is a *Yes*-instance of 3P. Let Γ_A be a fan-planar drawing of G_A preserving the rotation system $\mathcal{R}_A$. We first observe that the top beam and the bottom beam are disjoint, otherwise there would be at least a 2-hope edge in one beam that is crossed by another edge of the other beam, thus violating the fan-planarity. We also note that columns can partially cross each other, but this does not actually affect the validity of the proof. Indeed, an edge e of a column L might cross an edge e' of another column L' only if e is incident to a vertex in the rightmost (leftmost) side of L, e' is a leftmost (rightmost) vertical edge of L', and L and L' are two consecutive columns. With a similar argument, it is immediate to see that vertices u and v must be separated by all the columns. Therefore, every transversal path satisfies conditions R.1, R.2 and it must pass through at least three central cells, if not it would cross a number of pairwise disjoint edges that is greater than the number of its edges, hence Γ_A would not be fan-planar. On the other hand, because of condition R.4, which is obviously satisfied, there cannot be any transversal path passing through more than three central cells. Otherwise, there would be some other transversal path that traverses a number of central cells that is strictly less than three. Hence, also condition R.3 is satisfied. In conclusion, every transversal path π_j ($j \in \{1, 2, \dots, m\}$) crosses $(3m-3)K+B$ vertical edges and traverses exactly three central cells C_{1j}, C_{2j} and C_{3j}. If $m(C_{1j}), m(C_{2j})$ and $m(C_{3j})$ denote the number of edges of these cells, then $m(C_{1j}) + m(C_{2j}) + m(C_{3j}) = B$, because each non-central cell has K edges. Therefore, the partitioning of A defined by $A_1, A_2, \dots, A_m$, where $A_j = \{m(C_{1j}), m(C_{2j}), m(C_{3j})\}$, is a solution of 3P for the instance A. □

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