Assignment 12

Post Date: 21 Jan 2019   Due Date: 28 Jan 2019, 11:30
You are permitted and encouraged to work in groups of two.

Problem 1: MultiCut in Trees   4 Points

The problem MultiCut in Trees takes as input a tree $T = (V, E)$, a set $H \subseteq \binom{V}{2}$ of pairs of vertices, and a threshold $k$. It asks whether it is possible to separate the pairs in $H$ by removing at most $k$ edges from $T$, i.e., whether there is a set $E' \subseteq E$ of size at most $k$ such that $s$ and $t$ are in different connected components of $T - E := (V, E \setminus E')$ for any pair $(s, t) \in H$.

Show that MultiCut in Trees with the threshold $k$ as parameter is fixed-parameter tractable.

Problem 3: Covering Points with Lines   6 Points

The problem Line Cover is defined as follows:

Input: A set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$ in the plane.

Parameter: $k \in \mathbb{Z}_{\geq 0}$

Output: Is there a set $L$ of at most $k$ lines such that each point $p_i \in P$ lies on a line $L$?

(a) Consider the instance of Line Cover given on the right. Show that the points can be covered with 3 lines.

(b) Prove: Any line containing at least $k + 1$ points of $P$ must be included in the solution of Line Cover with parameter $k$.

(c) Show that Line Cover parameterized by $k$ is fixed-parameter tractable.
Problem 2: Solving the LP-Relaxation of Vertex Cover 10 Points

Show how to use a max-flow algorithm in order to find an optimum solution for the LP-relaxation of vertex cover. To this end consider the following steps for a graph $G = (V, E)$.

Let $G' = (A \cup B, E')$ be the bipartite graph that is constructed from $G$ as follows. For each vertex $v \in V$ there are two vertices $a_v \in A, b_v \in B$, and for each edge $\{u, w\} \in E$, there are the edges $\{a_u, b_w\}, \{a_w, b_u\} \in E'$.

(a) Let $V'$ be a minimum vertex cover of $G'$. Show that

$$x_v := \begin{cases} 1 & \text{if } \{a_v, b_v\} \subseteq V' \\ 0 & \text{if } \{a_v, b_v\} \cap V' = \emptyset \\ 0.5 & \text{else} \end{cases}$$

is an optimal solution to $G'$s LP relaxation of Vertex Cover.

Consider the following flow network $D$: Direct the edges of $G'$ from $A$ to $B$. Add a source $s$ with edges to all $a \in A$ and and a sink $t$ with edges from all $b \in B$ to $G'$. All capacities are one. Let $f$ be a maximum integer $s$-$t$-flow on $D$. Let $M = \{e \in E(A, B); f(e) > 0\}$.

(b) Show that $|V'| \geq |M|$ for any vertex cover $V'$ of $G'$.

Let $S = \{v \in V; \text{there is a directed } s-v\text{-path in } D_f\}$. Let $V' = (A \setminus S) \cup (B \cap S)$.

(c) Show that $V'$ is a vertex cover of $G'$.

(d) Show that each vertex in $V'$ is incident to an edge in $M$.

(e) Show that no edge in $M$ is incident to two vertices in $V'$. 