Problem 1: Venkatesan’s Approach 6 Points

Let \((D = (V, E), s, t, c)\) be a planar bidirected flow network. Choose a directed \(s-t\)-path \(P\) of \(D\). For \((v, w) \in V \times V\), set \(\pi(v, w) = 1\) if \((v, w) \in P\), \(\pi(v, w) = -1\) if \((w, v) \in P\), and \(\pi(v, w) = 0\) otherwise. Let \(\lambda \in \mathbb{R}_0^+\) be such that the directed dual graph \(D^*\) with edge length \(\ell_\lambda(e^*) = c(e) - \lambda \pi(e), \ e^* \in E^*\) does not contain a negative directed cycle, i.e., such that the shortest path distances \(d_\lambda(v^*, w^*)\) in \(D^*\) with respect to \(\ell_\lambda\) are well defined. Choose an arbitrary vertex \(s^*\) of \(D^*\). Prove that

\[
\phi_\lambda(e) = \max(0, d_\lambda(s^*, \text{right}(e)) - d_\lambda(s^*, \text{left}(e)) + \lambda \pi(e))
\]

is a flow in \(D\) with value \(\lambda\), i.e. show that the following properties are fulfilled:

(a) capacity constraint

(b) flow conservation, and

(c) \(w(\phi_\lambda(e)) = \lambda\).

Problem 2: Separators of Trees 4 Points

Let \(T\) be a tree with non-negative weights on the vertices that sum to one. A \textit{weighted vertex separator} of \(T\) is a partition of the vertex set into two sets \(A\) and \(B\) of weight at most \(2/3\) and a vertex \(v\) such that there is no edge between \(A\) and \(B\).

(a) Show how to compute a weighted vertex separator of a tree in linear time.

(b) Can the vertex set of any tree with non-negative weights on the vertices summing to one be partitioned into two sets \(A\) and \(B\) of weight at most \(1/2\) and a vertex \(v\) such that there is no edge between \(A\) and \(B\)?