Assignment 11

Post Date: 4 July 2014  Due Date: 11 July 2014  Tutorial: 18 July 2014

You are permitted and encouraged to work in groups of two.

Problem 1: Independent Vertex Sets 4 Points

Let $G = (V, E)$ be a graph. Let $I = \{V' \subseteq V; \{u, v\} \notin E \text{ for } u, v \in V'\}$.

(a) Show that $(V, I)$ is an independence system.

(b) Is $(V, I)$ also a matroid?

Problem 2: Matching Matroid 6 Points

Let $G = (V, E)$ be a graph. Let

$$I = \{V' \subseteq V; \text{there is a matching } M \text{ of } G \text{ s.t. no vertex in } V' \text{ is free}\}.$$  

Show that $(V, I)$ is a matroid.

Problem 3: Weight Sequence 4 Points

Let $(X, I)$ be a matroid and let $\omega : X \to \mathbb{R}$ be a weight function. The weight sequence of a basis $B = \{x_1, \ldots, x_d\}$ of $(X, I)$ is the sequence $\langle \omega(x_1), \ldots, \omega(x_k) \rangle$ of weights of the elements of $B$ ordered such that $\omega(x_1) \leq \cdots \leq \omega(x_k)$.

Show that any two minimum weight bases of a matroid have the same weight sequence.
Problem 4: Fundamental Cycle Basis

Let $G = (V, E)$ be an undirected, connected graph with $m$ edges. Let $T = (V, E_T)$ be a spanning tree of $G$. For each non-tree edge $e = \{v, w\} \in E \setminus E_T$ we define a fundamental cycle $C_e = \{e\} \cup P_e$ where $P_e$ is the set of edges on the unique path in $T$ between $v$ and $w$. Show that the fundamental cycle basis $B_T = \{C_e \mid e \in E \setminus E_T\}$ with respect to $T$ is indeed a cycle basis of the cycle space $\mathcal{C}$:

(a) Show that $B_T$ is linearly independent.

(b) Show that $B_T$ is a generating system of $\mathcal{C}$ by describing how an arbitrary simple cycle can be written as a linear combination of elements of $B_T$.

**Hint:** Consider the cuts of $G$ induced by the connected components of $T$ minus one edge. Observe that a cycle crosses a cut an even number of times.