Assignment 8

Post Date: 13 June 2014  Due Date: 20 June 2014  Tutorial: 25 June 2014
You are permitted and encouraged to work in groups of two.

Problem 1: Successive Shortest-Paths 5 Points

Consider successive shortest path on the equivalent min-cost max-flow network. Always perform a single source shortest path in the residual network regarding the reduced costs from the global source $s$. Always augment on a path with minimum costs in the residual network from $s$ to the global sink $t$.

Show that the following holds after each update of the potentials and before the following augmentation of the flow:

The vertex potentials always correspond to the shortest path distances from $s$ in the residual network regarding the original costs.

Problem 2: Matchings on Bipartite Graphs 7 Points

A minimum vertex cover of a graph $G = (V, E)$ is a set of nodes $C \subseteq V$ of minimal size, such that every edge $e \in E$ is incident to one of the nodes in $C$. Formally: $\forall (v, v') \in E : C \cap \{v, v'\} \neq \emptyset$.

(a) Show that a cardinality maximum matching on a bipartite graph has the same size as a minimum vertex cover. (Hint: Argument via the dual LP.)

(b) How about non-bipartite graphs?
Problem 3: Perfect Matchings

Let $T$ be a tree and $G = (A, B, E)$ be a bipartite graph.

(a) Show that $T$ has at most one perfect matching.

(b) Show that if $|A| = |B| = n$ and $|E| > (k - 1)n$ then there exists a matching $M$ on $G$ with $|M| \geq k$.

Let now $M$ be a matching on $G$. We transform $G$ in a directed network $D$ by directing an edge from $A$ to $B$ if it does not belong to $M$ and from $B$ to $A$ if it does belong to $M$.

(c) Show that there exists an augmenting path in $G$ with respect to $M$ if and only if there exists a directed path in $D$ between an unmatched vertex in $A$ and an unmatched vertex in $B$. 