Assignment 4

Available Since: May 11, 2011  Due Date: May 18, 2011, 2:30 p.m.
You are permitted and encouraged to work in groups of two.

Exercise 1: Exponential Area Requirement 6 Points

Let $G_n = (V_n, E_n), n \in \mathbb{N}$ be recursively defined as follows.

- $V_0 = \{s_0, t_0\}$ and $E_0 = \{(s_0, t_0)\}$.
- $V_n = V_{n-1} \cup \{s_n, t_n\}$ and $E_n = E_{n-1} \cup \{(t_{n-1}, t_n), (s_n, s_{n-1}), (s_{n-1}, t_n), (s_n, t_n)\}$.

Let the embedding of $G_n$ be such, that $(s_{n-1}, t_n)$ is on the right hand side and $(s_n, t_n)$ on the left hand side of $G_{n-1}$.

(a) Prove that $G_n$ is series-parallel.

(b) Consider an upward planar, embedding-preserving, straight-line drawing of $G_n$. Let $A(G_n)$ be the area of the triangle with vertices $s_n$, $t_n$, and $s_{n-1}$. Show that

$$A(G_n) \geq 4 \cdot A(G_{n-1}).$$

Hint: Consider the horizontal lines through $s_{n-1}$ and $t_{n-1}$, and the straight line through $s_{n-1}$ and $s_{n-2}$. Find out where $t_n$ and $s_n$ could possibly be drawn and consider then the straight line through $t_n$ and $t_{n-1}$.

(c) Conclude that $A(G_n) \in \Omega(4^n)$.
Exercise 2: Decomposition Tree 6 Points

Consider a directed multigraph. In the beginning each edge is ladled with a tree that consists of a single vertex labeled $Q$. Consider the following two types of reduction steps.

**Series reduction** Let $v$ be a vertex with in- and outdegree one. Let $(u, v)$ labeled $T_1$ and $(v, w)$ ladled $T_2$ be the edges that are incident to $v$. Replace $v$ and its incident edges by the single edge $(u, w)$ labeled with the tree that consists of a root labeled $S$ with left subtree $T_1$ and right subtree $T_2$.

**Parallel reduction** Replace two edges of the form $(u, v)$ labeled $T_1$ and $T_2$ by one edge of the form $(u, v)$ labeled with the tree that consists of a root labeled $P$ with left subtree $T_1$ and right subtree $T_2$.

(a) Show that if a directed multigraph is series-parallel then it can be reduced by a sequence of series and parallel reductions to a graph that contains one edge.

(b) Show that if a directed multigraph $G$ can be reduced by a sequence of series and parallel reductions to a graph that contains one edge than the label of this last edge is a decomposition tree of $G$. 